The Extremal Length Method

A geometric conformally invariant notion

Def. Extremal length.

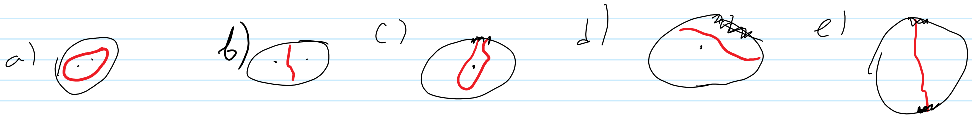
Γ - family of rectifiable piecewise connected curves in domain Ω.
$\rho \geq 0$ - measurable function in Ω (conformal metric).

$A(\rho) := \iint_\Omega \rho^2(z)\, dA(z)$ - the area of Ω wrt ρ.

$L(\rho) := \inf_{\gamma \in \Gamma} \int_\gamma \rho(z)|dz|$ - the shortest length in metric ρ.

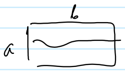
$\lambda(\Gamma) := \sup_\rho \frac{L^2(\rho)}{A(\rho)}$ — extremal length.

Examples. a) b) c) d) e)



Thm. (Main reason for introduction: invariance)
Let $f: \Omega \to \Omega'$ — conformal, $\Gamma' := \{f(\gamma) : \gamma \in \Gamma\}$. Then $\lambda(\Gamma') = \lambda(\Gamma)$

Proof. For ρ on Ω, define $\rho' := \frac{\rho}{|f'|} \circ f^{-1}$.
$A(\rho') = A(\rho)$, $L(\rho') = L(\rho)$ $\left(\int_{f(\gamma)} \rho'(w)|dw| = \int_\gamma \rho(z)|dz|\right)$, so
$\lambda(\Gamma') \geq \lambda(\Gamma)$. But $+ f^{-1}$.

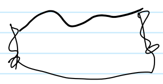
Examples: 1)  **Thm** $\lambda(\Gamma) = \frac{b}{a}$

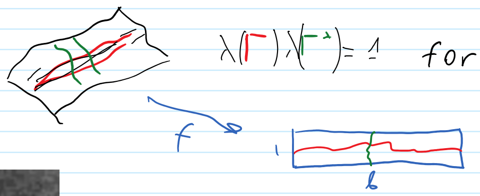
Pf Lower bound (easy): $\rho \equiv 1$ $L(\rho) = b$, $A(\rho) = ab \Rightarrow \lambda(\Gamma) \geq \frac{b^2}{ab} = \frac{b}{a}$.

Upper Bound. Normalize: $L_\rho(\Gamma) = 1 \Rightarrow \ell_\rho(\gamma) \geq 1 \ \forall \gamma \in \Gamma$.

Then $\int_0^a \int_0^b \rho^2\, dx\, dy \underset{\text{Cauchy}}{\geq} \int_0^a \frac{1}{b}\left(\int_0^b \rho(x,y)\, dx\right)^2 dy \geq \int_0^a \frac{1}{b}\, dy = \frac{a}{b}$, so $\frac{L(\rho)^2}{A(\rho)} \leq \frac{b}{a}$ ∎

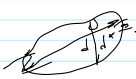
Corollary. $\exists$ Conformal map $f: a_1 \square^{b_1} \to a_2 \square^{b_2} \iff \frac{a_1}{b_1} = \frac{a_2}{b_2}$

Corollary (Conformal rectangles)  Ω, $\lambda_1, \lambda_2 \subset \partial\Omega$ - connected. Then
$\exists f: \Omega \to \{0 \le x \le b, 0 \le y \le 1\}$, with λ_1, λ_2 mapped to vertical sides, where
$b = \lambda(\Gamma)$, Γ - curves joining λ_1 to λ_2 in Ω. Def $d_\Omega(\lambda_1, \lambda_2)$ - extremal distance

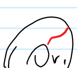
Corollary  $\lambda(\Gamma)\lambda(\Gamma^*) = 1$ for any conformal rectangle.

$\lambda(\Gamma) = b$
$\lambda(\Gamma^*) = \frac{1}{b}$

Oswald Teichmüller (1913-1943)

Corollary (Teichmuller Thm)  d - distance from λ_1 to λ_2. d^* - distance between complements. Then $d d^* \le A(\Omega)$

Pf. $\lambda(\Gamma) \ge \frac{d^2}{A(\Omega)}$
$\lambda(\Gamma^*) \ge \frac{(d^*)^2}{A(\Omega)} \Rightarrow 1 = \lambda(\Gamma)\lambda(\Gamma^*) \ge \left(\frac{d d^*}{A(\Omega)}\right)^2$ ∎

2)  $\Omega = \{r_1 < |z - z_0| < r_2\}$. Γ - curves joining inner and outer.
Thm. $\lambda(\Gamma) = \frac{1}{2\pi} \log \frac{r_2}{r_1}$.
Pf. $\rho(z) = \frac{1}{|z|}$ - lower bound. $A(\rho) = \int_{r_1}^{r_2}\int_0^{2\pi} \frac{r\,dr\,d\theta}{r^2} = 2\pi \log\frac{r_2}{r_1}$, $L(\rho) = \int_{r_1}^{r_2} \frac{dr}{r} = \log\frac{r_2}{r_1}$.
Upper bound: As before, normalize $L(\rho) = 1$, $\forall\theta$ $\int_{r_1}^{r_2} \rho(re^{i\theta})\,dr \ge 1$. Then
$1 \le \left(\int_0^{2\pi}\int \rho(re^{i\theta})\,dr\right)^2 \le \left(\int_0^{2\pi}\int_{r_1}^{r_2} \rho^2(re^{i\theta})\,r\,dr\,d\theta\right)\left(\int_0^{2\pi}\int_{r_1}^{r_2} \frac{r\,dr}{r^2}\right) = A(\rho)\log\frac{r_2}{r_1}$ ∎

Corollary. Ω - doubly connected domain. $\exists$ map $\Omega \to A_{r_1, r_2}$ where
$\frac{1}{2\pi}\log\frac{r_2}{r_1} = \lambda(\Gamma)$, where Γ - curves joining two components.

2')  $\lambda(\Gamma) = \frac{2\pi}{\log\frac{r_2}{r_1}}$ Same proof. So again, for any doubly connected domain, $\lambda(\Gamma)\lambda(\Gamma^*) = 1$.
Modulus of doubly connected domain.

2')  $\lambda(\Gamma) = \frac{2\pi}{\log \frac{r_2}{r_1}}$ Same proof. So annulus, ~~X~~ i.e. any doubly connected domain, $\lambda(\Gamma)\lambda(\Gamma^*) = 1$.

Modulus of doubly connected domain.

Some properties of Extremal Length.

Thm (Uniqueness). If ρ_1, ρ_2 extremal $A(\rho_1) = A(\rho_2) \Rightarrow \rho_1 = \rho_2$ a.e.

Pf. WLOG $A(\rho_1) = A(\rho_2) = 1$. Take $\rho_3 = \frac{1}{2}(\rho_1 + \rho_2)$. Then

$L(\Gamma, \rho_3) = \inf_\gamma \int \frac{1}{2}\rho_1 + \frac{1}{2}\rho_2 \geq \frac{1}{2}\inf_\gamma \int \rho_1 + \frac{1}{2}\inf_\gamma \int \rho_2 = \frac{1}{2}(L(\Gamma, \rho_1) + L(\Gamma, \rho_2)) = 1.$

$A(\Omega, \rho_3) = \frac{1}{4}A(\rho_1) + \frac{1}{4}A(\rho_2) + \frac{1}{2}\int \rho_1\rho_2 \underset{\text{Cauchy}}{\leq} 1$, equality $\Leftrightarrow$ $\rho_1 = \rho_2$ a.s. ∎

Remark Not always exist!

Rules for extremal length.

Rule 1. (Extension rule) Let $\Omega \subset \Omega'$, $\forall \gamma' \in \Gamma'\ \exists \gamma \in \Gamma: \gamma \subset \gamma'$.

Then. $\lambda_{\Omega'}(\Gamma') \geq \lambda_\Omega(\Gamma)$.

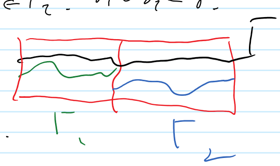
Pf. Consider ρ - (almost) extremal for Γ, & $\rho' = \begin{cases} \rho & \text{on } \Omega \\ 0, & \text{on } \Omega' \setminus \Omega \end{cases}$.

Then $A(\rho) = A(\rho')$, $L(\Gamma, \rho) \leq L(\Gamma', \rho')$

Rule 2. (Serial rule). Γ_1, Γ_2 - curve family in disjoint Ω_1, Ω_2 ($\Omega_1 \cap \Omega_2 = \emptyset$).

Γ - curve family in $\Omega_1 \cup \Omega_2 =: \Omega$, s.t. $\forall \gamma \in \Gamma\ \exists \gamma_1 \in \Gamma_1\ \gamma_2 \in \Gamma_2: \gamma_1 \cup \gamma_2 \subset \gamma$.

Then. $\lambda(\Gamma) \geq \lambda(\Gamma_1) + \lambda(\Gamma_2)$.



If $\lambda(\Gamma_1)$ or $\lambda(\Gamma_2) = 0$ or ∞, use Rule 1.

Otherwise, choose ρ_1, ρ_2: $A(\rho_1) = L(\rho_1)$, $A(\rho_2) = L(\rho_2)$.

$\rho := \rho_1 \chi_{\Omega_1} + \rho_2 \chi_{\Omega_2}$ - metric on Ω.

$L(\Gamma, \rho) \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$ $\quad A(\Omega, \rho) = A(\Omega_1, \rho_1) + A(\Omega_2, \rho_2) = L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$

So $\lambda(\Gamma) \geq \frac{L^2(\Gamma, \rho)}{A(\rho)} \geq L(\Gamma_1, \rho_1) + L(\Gamma_2, \rho_2)$ ∎

Take $\sup_{\rho_1, \rho_2}$

Rule 3. (Parallel rule). Ω_1, Ω_2 - disjoint, Γ_1 in Ω_1, Γ_2 - in Ω_2. Γ - in $\Omega \supseteq \Omega_1 \cup \Omega_2$

such that $\forall \gamma \in \Gamma_1 \cup \Gamma_2\ \exists \gamma' \in \Gamma$, $\gamma' \subset \gamma$. Then

$$\frac{1}{\lambda(\Gamma)} \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}.$$

Pf. Take ρ on Ω, normalize $L(\Gamma, \rho) = 1$. Then $L(\Gamma_1, \rho) \geq 1$, $L(\Gamma_2, \rho) \geq 1$.

$A(\rho) \geq A(\Omega_1, \rho) + A(\Omega_2, \rho) \geq \frac{1}{\lambda(\Gamma_1)} + \frac{1}{\lambda(\Gamma_2)}$. Take sup over ρ. ∎

Rule 4. (Symmetry rule) T - symmetry on Ω. $T: \Omega \to \Omega$ - analytic or antianalytic

$T \circ T = id$. Let $\Gamma \subset \Omega$, $T(\Gamma) = \Gamma$. Then

Rule 4. (Symmetry rule) T – symmetry on Ω. $T: \Omega \to \Omega$ – analytic or antianalytic $T \circ T = \mathrm{id}$. Let $\Gamma \subset \Omega$, $T(\Gamma) = \Gamma$. Then

$$\lambda(\Gamma) = \sup \left\{ \frac{L^2(\Gamma, \rho)}{A(\Omega, \rho)} : \rho = (\rho \circ T)\underbrace{|T'|}_{\text{analytic}} \text{ or } \rho = (\rho \circ T)\underbrace{|\overline{T}'|}_{\text{antianalytic}} \right\}$$

Pf. $\geq$ – obvious. more restricted family.

$\leq$ Take ρ – metric, $\rho' := (\rho \circ T)|T'|$ (or $|\overline{T}'|$) Then $L(\Gamma, \rho) = L(\Gamma, \rho_1)$, $A(\rho) = A(\rho_1)$.

so $\rho_2 = \frac{1}{2}(\rho_1 + \rho_2)$ – symmetric, satisfies, as above

$$\frac{L^2(\Gamma, \rho_2)}{A(\rho_2)} \geq \frac{L^2(\Gamma, \rho)}{A(\rho)}$$